J. Fluid Mech. (1976), *vol.* 77, *part* 3, *pp.* 561–579
Printed in Great Britain

Some notes on unsteady lifting-line theory

By TH. VAN HOLTEN
Delft University of Technology, Department of
Aerospace Engineering, Delft, Netherlands

(Received 28 November 1975)

A higher-order lifting-line theory for a wing in unsteady motion is discussed. Apart from the addition of higher-order terms, it also differs from the theory derived by James (1975) in its emphasis on 'physical' interpretations. This emphasis has made it possible to shed some new light on Prandtl's classical lifting-line theory, as well as on Weissinger's $\frac{3}{4}$-chord theory.

1. Introduction

Recently a lifting-line theory for the unsteady wing problem was published by James (1975). This theory was derived using a matched asymptotic expansion method.

The subject of unsteady lifting-line theory was in recent years also investigated by the author of the present paper, using a similar matched asymptotic expansion method. This particular work was related to the analysis of the pressure distribution on the blades of a helicopter in forward flight. A higher-order lifting-line method was developed and published (Van Holten 1974, 1975*a*) which rigorously takes into account all the unsteady and yawed flow phenomena of inviscid theory encountered by the blades of a helicopter rotor.

On comparing the papers written by James and the present author, it is interesting to see how the two approaches, although treating essentially the same problem by essentially the same kind of mathematical technique, nevertheless differ in many respects, especially in the formulating of the results. This is largely due to the fact that the present author, in accordance with the applied nature of the problem he considered, used in his papers as many physical arguments and interpretations as he could. In fact, a recent paper (Van Holten 1975*b*) was almost entirely devoted to a discussion of analytical models, concepts and practices in present day rotor analysis, critically examined in the light of the more rigorous asymptotic theory. This required a 'translation' of the asymptotic theory and its results into the terminology of conventional applied aerodynamics.

The purpose of the present paper, too, is to consider unsteady lifting-line theory from a 'physical' rather than a formal point of view. It is thus complementary to James's paper. The complete theory of the helicopter blade will not be given here – it has been published elsewhere – and the following discussion is limited to only the simplest 'model' situations, in order not to burden the discussion with non-essential details.

In one area the present paper touches a subject not covered by James's paper. This is the higher-order lifting-line theory and the light it sheds on Weissinger's well-known $\frac{3}{4}$-chord method.

2. The linearized boundary-value problem for the uncambered rectangular wing

As a preliminary we shall consider an uncambered rectangular wing placed in a steady parallel flow perpendicular to the span (figure 1). We assume incompressible flow and small perturbations, so that the following linearized flow equations apply:

$$\partial \mathbf{V}/\partial t + (\mathbf{U}\,.\,\nabla)\,\mathbf{V} = -\rho^{-1}\,\mathrm{grad}\,p \quad \text{(Euler's equation)}, \tag{1}$$

$$\mathrm{div}\;\mathbf{V} = 0 \quad \text{(continuity equation)}, \tag{2}$$

where $\mathbf{U} = \mathbf{i}U$ is the free-stream velocity, $\mathbf{V}$ is the velocity perturbation, with components ($\mathbf{i}u$, $\mathbf{j}v$, $\mathbf{k}w$), and p denotes the pressure perturbation. Taking the divergence of (1) and substituting (2) yields the Laplace equation for p:

$$\nabla^2 p = 0. \tag{3}$$

In terms of the Cartesian co-ordinates (x, y, z) the following boundary-value problem results for the rectangular wing considered. The pressure perturbation should satisfy Laplace's equation:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2} = 0. \tag{4}$$

The pressure perturbation should vanish at large distances from the wing:

$$p \to 0 \quad \text{for} \quad x^2+y^2+z^2 \to \infty. \tag{5}$$

According to (1) the normal component of the pressure gradient on the wing surface should vanish in the case of an uncambered wing:

$$\partial p/\partial y = 0 \quad \text{on the wing surface.} \tag{6}$$

In linearized theory this boundary condition is applied to the part of the x,z plane for which $-\frac{1}{2}c \leqslant x \leqslant \frac{1}{2}c$ and $-\frac{1}{2}b \leqslant z \leqslant \frac{1}{2}b$, with c and b denoting the chord length and span respectively. Along the leading edge of the wing there will in general exist a streamline kink, which implies a pressure singularity:

$$p \to -\infty \quad \text{along the leading edge.} \tag{7}$$

The strength of this singularity should be such that the flow becomes tangential to the wing surface. Because it has already been required by (6) that the curvature of the flow on the surface is correct, it is sufficient to require the flow to be tangential along one line on the wing only. A convenient choice is the mid-chord line:

$$v(0,0,z)/U = -\alpha(z) \quad \text{along the mid-chord line,} \tag{8}$$

where $\alpha(z)$ is the geometrical angle of attack of the wing chords with respect to the x, z plane.

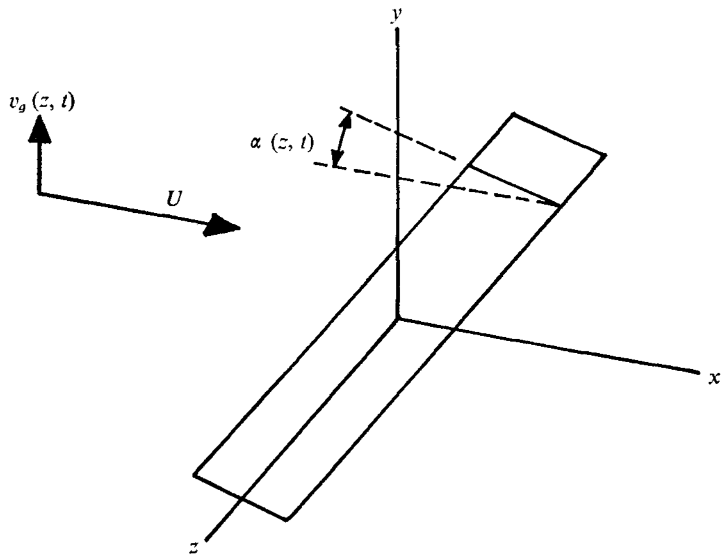

FIGURE 1. Notation for rectangular wing.

The Kutta condition, requiring smooth flow at the trailing edge, is implied by the above boundary-value problem, because a streamline kink is allowed to occur along the leading edge only. It will sometimes be convenient to use the fact that the pressure field must be antisymmetric with respect to the x, z plane, which is also implicit in the boundary-value problem.

3. Asymptotic approximation of the near field

In order to derive a lifting-line theory, the following physical assumption is needed: *the variations in the pressure in the spanwise direction have a characteristic length of the order of the span, whereas the variations in the pressure in the chordwise direction have a characteristic length of the order of the chord.* Evidently, this assumption can be valid only in the so-called near field of the wing, i.e. the field close to the wing surface, excluding the regions near the tips. Rewritten in terms of the characteristic co-ordinates x/c, y/c and z/b, Laplace's equation reads

$$\frac{\partial^2 p}{\partial(x/c)^2}+\frac{\partial^2 p}{\partial(y/c)^2} = -\frac{1}{A^2}\frac{\partial^2 p}{\partial(z/b)^2}, \tag{9}$$

where A is the aspect ratio b/c. On account of the physical assumption mentioned above, the partial derivatives in (9) are all of the same order of magnitude.

It follows immediately from (9) that p satisfies a two-dimensional Laplace equation when A is very large ($A\to\infty$). One may refine the analysis by writing the near pressure field in the following form:

$$p = p_0+\frac{1}{A}p_1+\frac{1}{A^2}p_2+\ldots \quad \text{for} \quad A\to\infty. \tag{10}$$

This is an asymptotic expression, in which the first term is the two-dimensional pressure field, whereas the other terms describe the way in which the pressure field becomes two-dimensional when the aspect ratio grows larger and larger. It will appear later that terms behaving like $A^{-2}\ln A$ for $A \to \infty$ also occur. For convenience, such terms have not been written explicitly in (10), but are assumed to be included in the corresponding term having an asymptotic behaviour like A^{-2}. Substituting the assumed type of solution (10) into (9) and equating terms of equal order, one arrives at the following conclusion: even when terms $O(A^{-1})$ are included, the pressure field still satisfies the two-dimensional Laplace equation

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 = 0 \quad \text{up to } O(A^{-1}). \tag{11}$$

In the higher-order approximation, where p_2 is also included, the near pressure field p satisfies a two-dimensional Poisson equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\frac{\partial^2}{\partial z^2}(p_{\text{two-dim}}) \quad \text{up to } O(A^{-2}), \tag{12}$$

where $p_{\text{two-dim}}$ is the solution obtained from (11). The discussion of the higher-order theory will be postponed until § 9 and the analysis in §§ 3–8 includes only terms up to $O(A^{-1})$. It is convenient to introduce elliptic co-ordinates, sketched in figure 2 and conforming to the transformation formulae

$$x = \tfrac{1}{2}c\cosh\eta\cos\phi, \quad y = \tfrac{1}{2}c\sinh\eta\sin\phi. \tag{13}$$

The value of η ranges between 0 and ∞, with η = constant representing ellipses degenerating for $\eta = 0$ into the wing chord. The co-ordinate ϕ ranges between $-\pi$ and $+\pi$, the lines ϕ = constant representing hyperbolas orthogonal to the ellipses. In terms of the elliptic co-ordinates, Laplace's equation (11) reads (see, for example, Moon & Spencer 1961)

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{(\frac{1}{2}c)^2(\cosh^2\eta - \cos^2\phi)}\left\{\frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \phi^2}\right\} = 0. \tag{14}$$

The boundary conditions (6) and (7) now read

$$\partial p/\partial\eta = 0 \quad \text{for} \quad \eta = 0 \tag{15}$$

except at the leading edge, where

$$p \to -\infty \quad \text{for} \quad \eta = 0, \quad \phi = \pm\pi. \tag{16}$$

The general antisymmetric solution of (14) satisfying the conditions (15) and (16) is given by

$$\frac{p}{\frac{1}{2}\rho U^2} = -\frac{C_{l_1}(z)}{\pi}\frac{\sin\phi}{\cosh\eta + \cos\phi} + \sum_{n=1}^{\infty} a_n(z)\cosh(n\eta)\sin(n\phi), \tag{17}$$

where the occurrence of the terms with $a_n(z)$, although these terms become infinite at large distances from the chord ($\eta \to \infty$), cannot be ruled out immediately: the present approximation for the pressure field is valid only in the near field, so that there is no condition at infinity. If it should emerge that these terms must be kept (and, as will be seen later, this is so in the higher-order

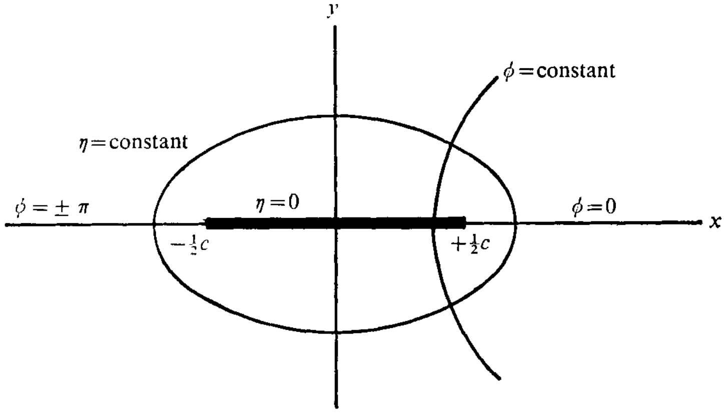

FIGURE 2. Elliptical co-ordinates.

theory!) they would contribute to the total sectional lift. Therefore, the coefficient of the first term on the right-hand side of (17), which is the field of a two-dimensional flat-plate aerofoil, is written as C_{l_1}. The index indicates that this might only be a part of the total lift coefficient of the wing section.

4. Asymptotic approximation of the far field

The far field is defined as the field at distances of the order of the wing span b from the wing surface. In this region the physical assumptions for the near field are no longer valid because the characteristic length scale of the far pressure field will be equal to the span b in all directions. However, looking at our problem on the scale of the span, another simplification may be introduced by noting that the limit $A = b/c \to \infty$ means that the chord length c shrinks to zero. This means that the far field in the asymptotic approximation will become the field of a line along which pressure singularities are distributed.

Such a field may be expressed in several different ways. For the purpose of actual calculations it has been found particularly useful to express the far field in terms of prolate spheroidal co-ordinates (ψ, θ, χ) (figure 3) defined by the transformation formulae

$$r = \tfrac{1}{2}b\sinh\psi\sin\theta, \quad \chi = \chi, \quad z = \tfrac{1}{2}b\cosh\psi\cos\theta. \tag{18}$$

Surfaces of constant ψ are ellipsoids, with $\psi = 0$ representing the lifting line. The surfaces of constant θ are hyperboloids orthogonal to the surfaces ψ = constant. A dipole distribution having a pressure field antisymmetric with respect to the x, z plane is, for instance, given by

$$\frac{p_{\mathrm{dip}}}{\frac{1}{2}\rho U^2} = \frac{\sin\chi}{2\pi}\sum_{n=1}^{\infty} A_n P_n^1(\cos\theta)\,Q_n^1(\cos\psi), \tag{19}$$

where P_n^1 and Q_n^1 denote associated Legendre functions of the first and second kind respectively. The advantages of this type of representation of the far field

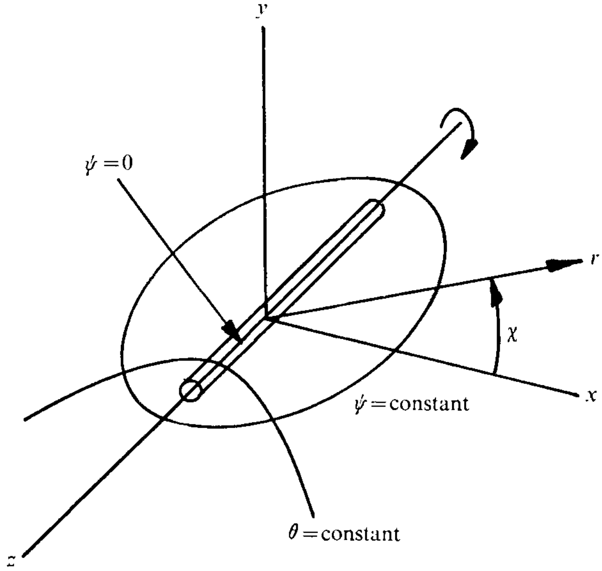

FIGURE 3. Prolate spheroidal co-ordinates.

will be seen later, in § 8. The behaviour of the field (19) close to the dipole line is

$$\frac{p_{\text{dip}}}{\frac{1}{2}\rho U^2} \sim -\frac{\sin\chi}{2\pi}\frac{\frac{1}{2}b}{r}\left\{1-\left(\frac{z}{\frac{1}{2}b}\right)^2\right\}^{\frac{1}{2}} \sum_{n=1}^{\infty} A_n P_n^1\left(\frac{z}{\frac{1}{2}b}\right) \quad \text{for} \quad r \to 0, \tag{20}$$

which expression enables us to determine the coefficients A_n if the dipole strength distribution is given.

For the purpose of purely analytical manipulations such as are needed in the present paper, a far-field representation in terms of the cylindrical co-ordinates (r, χ, z) (figure 3) involving modified Bessel functions has been found convenient. For this reason the analysis in the appendix uses the latter type of representation.

5. The matching condition

Neither the near nor the far field is uniquely determined as yet. In order to remove the indeterminacy the so-called 'matching condition' is needed. Using a physical argument, this condition will be derived here in a slightly different form from usual. Although the 'physical' discussion is perhaps somewhat less rigorous than the analysis due to Kaplun (Lagerstrom 1967), the final result will be shown to be equivalent to the usual matching condition, whereas the proposed interpretation of the matching condition has the advantage of eliminating the need to use a bit of 'mysticism' called the 'principle of least singularity' (Van Dyke 1964, p. 53).

The argument, in contrast to the usual asymptotic discussions which strictly focus on the limiting case $A \to \infty$, is based explicitly on the point of view that we in practice are really interested in the case of finite aspect ratio A. For both the near and far field of the wing asymptotic representations have been found, strictly valid only in the asymptotic limit $A \to \infty$. In practice, however, because the asymptotic results are always applied to cases of finite A, the near and far field are just regarded as approximations found in a convenient way. No such approximation can be found for the intermediate region: the physical assumptions leading to the simplification of the field equations close to the wing surface cannot be valid in the intermediate region, and neither is the degeneration of the boundary contour into a line valid there. In other words, for finite A one cannot assume the existence of an 'overlap' region where both approximations are simultaneously valid. The only way, then, to find an approximation for the complete pressure field is to find a suitable interpolation expression which bridges the gap between near and far field and smoothly merges with the approximate solutions of the near and far field. Such a uniformly 'valid' field may be found by summing the near and far pressure field, and subtracting a correction field, which will be called the 'common field'. The correction field must be chosen such that far from the wing it cancels the near field to the required order of accuracy so that only the far field remains. Close to the wing surface, the correction field has to cancel the far field, so that only the near field remains there. Denoting the complete pressure field thus obtained by $p_{\text{composite}}$, we have the structure

$$p_{\text{composite}} = p_{\text{near}} + p_{\text{far}} - p_{\text{common}}, \tag{21}$$

whereas it should be required that

$$p_{\text{common}} \sim (p_{\text{near}})_{r \to \text{order } b}, \tag{22}$$

and also

$$p_{\text{common}} \sim (p_{\text{far}})_{r \to \text{order } c}, \tag{23}$$

where the symbol $\sim$ denotes identity up to a certain specified order of accuracy. Combining (22) and (23) yields the matching condition:

$$(p_{\text{near}})_{r \to \text{order } b} \sim (p_{\text{far}})_{r \to \text{order } c}. \tag{24}$$

The crucial assumption in the argument is that a pressure field p_{common} can indeed be found which satisfies (22) and (23) simultaneously. The assumption is equivalent to the assumption that p_{near} and p_{far} can satisfy (24). Now p_{near} and thus the expression $(p_{\text{near}})_{r \to \text{order } b}$ satisfy (11) or (12). But (11) and (12) are also satisfied by the expression $(p_{\text{far}})_{r \to \text{order } c}$, because the physical assumptions leading to (11) and (12) apply just as well to a wing whose chord becomes vanishingly small ($c \to 0$). Therefore, $(p_{\text{near}})_{r \to \text{order } b}$ and $(p_{\text{far}})_{r \to \text{order } c}$ satisfy the same differential equation, and it is always possible to apply condition (24).

The usual matching condition is recovered by rewriting (24) in terms of the characteristic co-ordinates (r/c for the near field and r/b for the far field) and taking the asymptotic limit $A \to \infty$:

$$\lim_{r/c \to \infty} p_{\text{near}} \sim \lim_{r/b \to 0} p_{\text{far}}. \tag{25}$$

If one aims at a multiplicative structure of the composite field, the argument is almost identical. It is to be noted that nowhere in §§ 3–5 have the terms 'outer limit' and 'inner limit' been used. This terminology was avoided in order to emphasize that we are interested only in the case of finite aspect ratio, in which case the outer limit is just an asymptotic approximation for the far field and the inner limit is an approximation for the near field.

6. Application of the matching condition

At large distances from a wing section the elliptic co-ordinates (η, ϕ) and the circular-cylinder co-ordinates (r, χ) are related by

$$\phi = \chi + \tfrac{1}{16}(c/r)^2 \sin 2\chi + \ldots \quad (r/c \to \infty), \tag{26}$$

$$\eta = \ln(r/\tfrac{1}{4}c) - \tfrac{1}{16}(c/r)^2 \cos 2\chi + \ldots \quad (r/c \to \infty), \tag{27}$$

showing that far from the wing section the near field (17) behaves like

$$\frac{p_{\text{near}}}{\frac{1}{2}\rho U^2} = -\frac{C_{l_1}(z)}{2\pi}\frac{\sin\chi}{r/c} + \frac{C_{l_1}(z)}{8\pi}\frac{\sin 2\chi}{(r/c)^2} + \ldots + a_1(z)\frac{r}{\frac{1}{2}c}\sin\chi + \ldots \quad (r/c \to \infty). \tag{28}$$

For r of order b, i.e. on substituting $r = \beta b$ with $\beta = O(1)$, it is seen that the second term in (28) has an asymptotic behaviour $O(A^{-2})$ for $A \to \infty$. In the present approximation, where we neglect terms of such orders and smaller, we may thus simplify (28) to

$$\left(\frac{p_{\text{near}}}{\frac{1}{2}\rho U^2}\right)_{r \to \text{order } b} = -\frac{C_{l_1}(z)}{2\pi}\frac{\sin\chi}{r/c} + a_1(z)\frac{r}{\frac{1}{2}c}\sin\chi + \ldots, \tag{29}$$

where the terms with a_n must still be kept, since the asymptotic order of these coefficients is *a priori* not known. Nevertheless, (29) shows that to the present order of approximation the behaviour of the near field at distances of order b involves just one type of singularity, viz. a dipole of strength $C_{l_1}(z)c$, associated with the first term in the right-hand side of (29). According to (24), the far field thus consists of a discrete dipole line only. Formulae for the field of a dipole line are given in the appendix, where it is shown [equation (A5)] that the behaviour of the far field close to the line is given by

$$\frac{p_{\text{far}}}{\frac{1}{2}\rho U^2} = -\frac{C_{l_1}(z)}{2\pi A}\frac{\sin\chi}{r/b} + \frac{C_l''(z)}{\pi A}\sin\chi \frac{r}{b}\ln\left(\frac{r}{b}\right) + O\left(\frac{1}{A}\frac{r}{b}\right) \quad (r/b \to 0), \tag{30}$$

where C_{l_1}'' denotes the second spanwise derivative of C_{l_1} with respect to the non-dimensional co-ordinate $z/\frac{1}{2}b$. Again using the physical interpretation (24) of the matching condition, it is seen that for r of the order c, i.e. on substituting $r = \gamma c$ with $\gamma = O(1)$, the second term on the right-hand side of (30) is of order $A^{-2}\ln A$ for $A \to \infty$, and may be neglected. Using the matching condition (24) thus shows that $a_n = 0$ in (29), and the composite pressure field of the uncambered rectangular wing finally becomes

$$\frac{p}{\frac{1}{2}\rho U^2} = -\frac{C_l(z)}{\pi}\frac{\sin\phi}{\cosh\eta + \cos\phi} + \frac{p_{\text{dip}}}{\frac{1}{2}\rho U^2} + C_l(z)c\frac{\sin\chi}{2\pi r} \tag{31}$$

to $O(A^{-1})$.

7. Classical lifting-line theory

Having obtained an expression for the complete pressure field of the wing, we can calculate the field of velocity perturbations using (1). The evaluation of the velocity at time t_0 at a given point is equivalent to the computation of the velocity acquired by a particle of air coming from infinity upstream, travelling through the known pressure field and passing the point considered at the required time t_0. In the linearized theory the convective part of the acceleration in (1) is taken in the direction of the free-stream velocity $\mathbf{U}$ so that during the velocity integration the trajectory of the particle may be approximated by a straight line parallel to $\mathbf{U}$. The position of the particle relative to the wing is thus known at any instant of time, as well as the pressure gradient 'experienced' by the particle. The vertical velocity perturbation at a point z on the mid-chord line at time t_0 is thus found by evaluating the integral

$$\frac{v}{U}(0,0,z,t_0) = -\frac{1}{\rho U}\int_{-\infty}^{t_0} \frac{\partial p}{\partial y}\{x(t),0,z,t\}\,dt, \tag{32}$$

where $-\rho^{-1}\partial p/\partial y$ is the vertical acceleration experienced by the particle when it moves through the pressure field of the wing. In the case of steady flow the value of the integral will of course be independent of the time t_0, whereas $\partial p/\partial y$ depends on t only via the x co-ordinate of the particle. The more general formulation (32) is used here because it is also applicable to the unsteady cases considered later.

The function $\partial p/\partial y$ in (32) may be obtained from (31). Now the first term on the right-hand side of (31) is the pressure field of a two-dimensional flat-plate aerofoil. Consequently, this term contributes a velocity $v/U = -[C_l(z)]/2\pi$ as in two-dimensional aerofoil theory. The contribution of the other two terms on the right-hand side of (31) is symbolically written as $-v_i/U$.

The 'induced angle of attack' v_i/U is regarded in vortex theory as the contribution of the trailing vorticity of a lifting vortex line. In the pressure theory it is the contribution of the lifting pressure dipole line, *together with* the common term, consisting of a two-dimensional pressure dipole.

We can now apply the last remaining boundary condition, i.e. the tangency condition (8), in order to obtain an integral equation for the function $C_l(z)$:

$$\frac{v}{U}(0,0,z) = \left(\frac{v}{U}\right)_{\text{near field}} + \left(\frac{v}{U}\right)_{\substack{\text{far field+}\\ \text{common part}}} = -\frac{C_l(z)}{2\pi} - \frac{v_i}{U}(z) = -\alpha(z), \tag{33}$$

or, rewritten,

$$C_l(z) = 2\pi\{\alpha(z) - v_i(z)/U\}, \tag{34}$$

which is Prandtl's classical integral equation, stating that a wing section behaves like a two-dimensional aerofoil placed at an effective angle of attack $\alpha - v_i/U$.

As pointed out by Van Dyke (1964, p. 172) and James (1975), one can solve this equation by performing quadratures instead of solving it as an integral equation. This is so because the asymptotic theory indicates that v_i represents an effect $O(A^{-1})$. The dipole distribution causing v_i thus need not be known to a higher accuracy than $O(A^0)$ and one may take the strength of the distributed dipoles as predicted by a simple strip analysis, where every wing section is

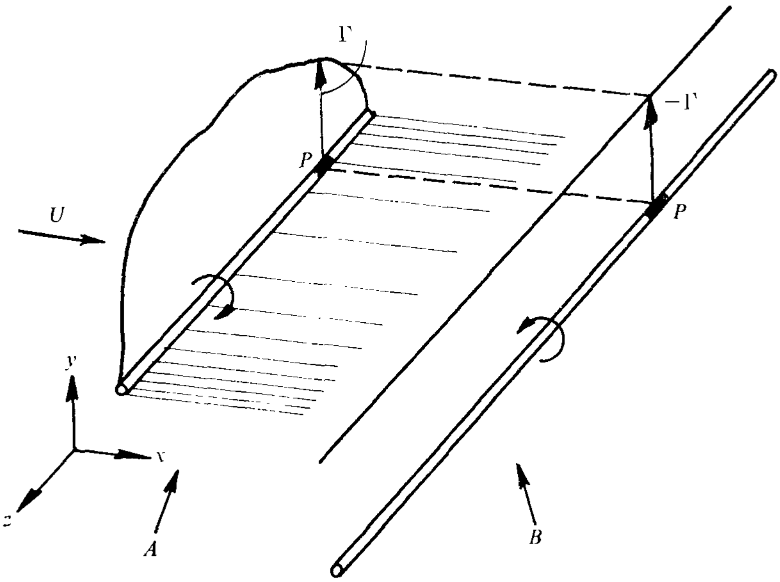

FIGURE 4. Definition of induced velocity at P: sum of contributions of vortex systems A and B.

assumed to be an isolated two-dimensional aerofoil. In practice, however, this leads to difficulties near the wing tips (figure 9·4 in Van Dyke 1964) and the formulation as an integral equation may often be preferred in actual calculations.

The present asymptotic analysis also points out a shortcoming in Prandtl's classical model. For, in the asymptotic approach to lifting-line theory v_i was found as the contribution of the pressure dipole line *together* with the common term. Translated into vortex terminology (figure 4), this means that v_i is the velocity at a point P due to the lifting vortex and its associated trailing vorticity *together* with the velocity due to a two-dimensional vortex of equal local strength but with opposite direction passing through the same point P. Naturally, this does not affect the quantitative results in steady flow: the contribution of the two-dimensional vortex to v_i is zero.

Things are very different, however, when we come to consider unsteady flow.

8. Unsteady lifting-line theory

In the case of unsteady lift, we should again take for $v_i(z, t)$ at the point P the velocity due to the lifting vortex line (having a wake of trailing as well as shed vorticity) *and add to this* the velocity due to the two-dimensional vortex, which is now also accompanied by shed vorticity (figure 5). Again, the strength of this two-dimensional vortex varies in time in the same way as the bound vorticity at P, although it is of opposite direction. It will be clear that this definition of induced velocity does *not* lead to infinite values of $v_i(z, t)$. One of the obstacles preventing the use of classical lifting-line concepts in unsteady flow has thus been traced back to a wrong interpretation of Prandtl's steady flow model.

The actual numerical or possibly analytical calculation of the induced velocity

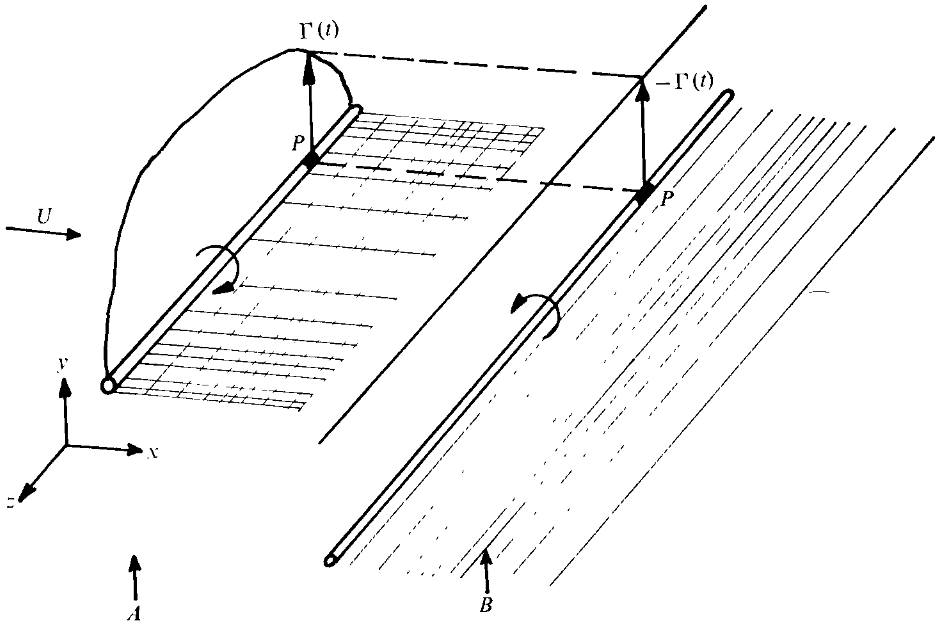

FIGURE 5. Definition of induced velocity at P; sum of contributions of vortex systems A and B.

in the case of unsteady lift is straightforward, using the definition obtained by combining (31) and (32):

$$\frac{v_i}{U}(0,0,z,t_0) = -\frac{1}{\rho U}\int_{-\infty}^{t_0} \frac{1}{r}\frac{\partial}{\partial \chi}\left\{p_{\mathrm{dlp}}(r,\chi,z,t)+l(z,t)\frac{\sin\chi}{2\pi r}\right\}_{\chi=\pi} dt, \qquad (35)$$

where in the integrand $r = U(t_0-t)$, and z is independent of time in the case of a steady parallel free-stream velocity perpendicular to the span. The symbol p_{dlp} again denotes a harmonic pressure field behaving near the lifting line like

$$-l(z,t)\,(\sin\chi/2\pi r). \qquad (36)$$

The expansion (30) clearly shows that the integrand of (35) near the lifting line only contains a logarithmic singularity, which is integrable, without having to attach a special definition to the integral of v_i. This is in contrast to James's formulation, where v_i is defined by a Hadamard integral.

If p_{dlp} is expressed using (19), the integrand has a form which proves to be very convenient for actual numerical calculations: in this way v_i is found for a number of orthogonal lift distributions by performing one-dimensional integrations with respect to time. This procedure replaces the two-dimensional spatial integration over the vortex sheet needed when Biot & Savart's theorem is used for calculating the induced velocity. It is furthermore illustrated in Van Holten (1975*a*) that the evaluation of v_i can always be reduced to a one-dimensional integral, no matter how complicated the wing motion is: the above assumption of a steady free stream is non-essential for the procedure.

Having obtained a rigorous and convenient definition of the induced velocity

v_i for unsteady lift, we can write down the integral equation for the time-dependent function $C_l(z, t)$. Let us assume that the rectangular wing considered is moving through a gust field, whose vertical velocity v_g in the x, z plane (figure 1) is $v_g(x, z, t)$.

Instantaneously the boundary-value problem is identical to the problem formulated in (4)–(8), provided the gust velocities are small enough to be considered as small perturbations. Consequently, the pressure field of the wing has the same form as the pressure field (31), except that C_l is a function of time:

$$\frac{p}{\frac{1}{2}\rho U^2} = -\frac{C_l(z,t)}{\pi}\frac{\sin\phi}{\cosh\eta+\cos\phi}+\frac{p_{\mathrm{dip}}}{\frac{1}{2}\rho U^2}(r,\chi,z,t)+C_l(z,t)\,c\frac{\sin\chi}{2\pi r}. \tag{37}$$

It should be carefully noted that this pressure field is entirely different from the field of a wing placed in a steady parallel flow where the unsteadiness results from a pitching or heaving motion of the wing with respect to an inertial frame of reference. In the latter case the motion of the wing surface implies a vertical acceleration of the particles of air moving along the wing surface, so that the near pressure field in (37) would have to be supplemented by an additional field taking care of this additional acceleration. In expression (37) the near field is the pressure field of a two-dimensional flat-plate aerofoil at rest with respect to an inertial frame of reference, although its lift is variable. The value of $v(z,t)/U$ along the mid-chord line contributed by the near pressure field should be calculated according to the two-dimensional theory for an aerofoil in a gust field and is symbolically written as

$$\left\{\frac{v}{U}(z,t)\right\}_{\text{near field}} = f_{\mathrm{2D,\,gust}}\{C_l(z,t)\}. \tag{38}$$

Applying the tangency condition gives

$$f_{\mathrm{2D,\,gust}}\{C_l(z,t)\}-\frac{v_i}{U}(z,t)+\frac{v_g}{U}(0,z,t) = -\alpha(z). \tag{39}$$

This yields, on inverting (39),

$$C_l(z,t) = C_{l_{\mathrm{2D,\,gust}}}\left\{\alpha(z)+\frac{v_g}{U}(0,z,t)-\frac{v_i}{U}(z,t)\right\}, \tag{40}$$

where $C_{l_{\mathrm{2D,\,gust}}}$ denotes the functional relationship between the time-varying gust angle v_g/U at the mid-chord point of a stationary two-dimensional aerofoil and its time-varying lift (see e.g. Fung 1968). It is seen that the wing sections may be analysed using the two-dimensional theory for an aerofoil in a gust field, using an *effective* gust field.

If α in (40) is a function of time $\alpha(z,t)$, then we have the case of a wing in pitching motion, and (40) does not remain valid. The pressure field (37) should then be supplemented with the field of a pitching two-dimensional aerofoil and the tangency condition should be modified accordingly. It is easily checked that the wing sections may then be considered to behave like two-dimensional pitching aerofoils, whereas the induced downwash associated with the lift due to pitching

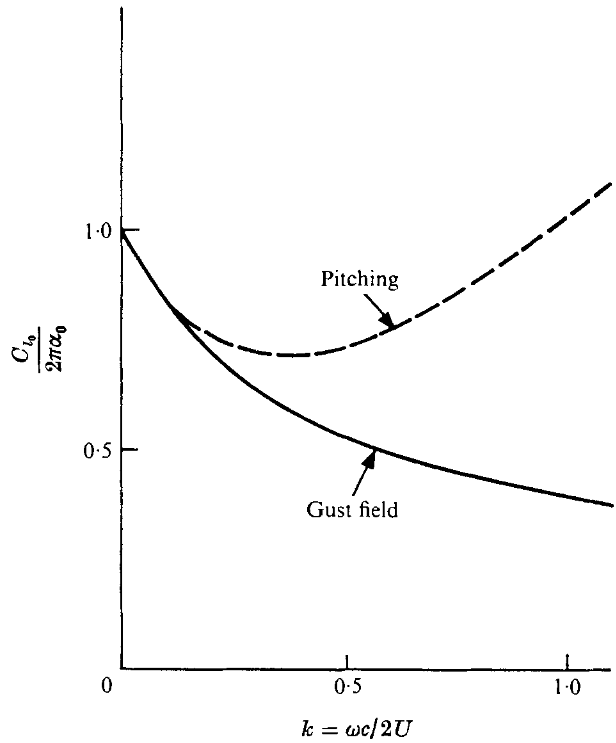

FIGURE 6. Amplitude of lift variation *vs.* reduced frequency of an aerofoil in pitching motion and in oscillating flow.

may be considered as a 'self-induced' gust field which adds to v_g. The unsteady lifting-line theory thus takes the form

$$C_l(z,t) = C_{l_{2D,\text{ pitching}}}\{\alpha(z,t)\} + C_{l_{2D,\text{ gust}}}\left\{\frac{v_g}{U}(0,z,t) - \frac{v_i}{U}(z,t)\right\}, \tag{41}$$

where $C_{l_{2D,\text{ pitching}}}$ denotes the functional relationship between the time-varying angle of incidence and the time-varying lift, as calculated using the two-dimensional theory of pitching aerofoils (see, for example, Fung 1968). The induced velocity v_i is the velocity caused by the *total* lift of the wing sections, i.e. the lift due to pitching as well as the lift due to the effective gust velocities.

Obviously, an analogous expression can be derived for the case of a heaving motion.

James (1975) finds for harmonic pitching motion $\alpha(z,t) = \alpha_0(z)\sin\omega t$ that lifting-line theory degenerates into strip analysis when the reduced frequency $k = \omega c/2U$ grows large. This result may be understood easily by considering figure 6. Two flat-plate aerofoils are compared, one oscillating around its $\frac{1}{4}c$ point, the other one at rest, but placed in a gust field.

In both cases it is assumed that the angle between the chord and the free-stream velocity varies like

$$\alpha(t) = \alpha_0 \sin\omega t. \tag{42}$$

The lift coefficient $C_l(t)$ varies like

$$C_l(t) = C_{l_0} \sin(\omega t + \phi), \tag{43}$$

where the amplitude C_{l_0} would have the value $C_{l_0} = 2\pi\alpha_0$ in quasi-steady conditions. Figure 6 shows the actual unsteady value of C_{l_0} as a function of the reduced frequency k, calculated by two-dimensional theory. It is seen that C_{l_0} for the pitching aerofoil increases for large reduced frequencies k whereas C_{l_0} in the gust case decreases steadily with increasing k. Looking back at (41) this shows that the three-dimensional effects on a pitching wing diminish in relative importance as k increases because $v_i(z,t)$ influences the sectional lift via a 'gust relation'. However, it may also be concluded from figure 6 that strip analysis is a rather crude device in the usual range of reduced frequencies $k = 0{\cdot}1$ to 1.

9. Higher-order lifting-line theory

As mentioned previously, owing to the incorrect interpretation of Prandtl's lifting-line model for steady flow, unsteady lifting-line theory has always met the problem of singular values of the induced downwash. Often one has tried to avoid this problem by using Weissinger's $\frac{3}{4}$-chord method, where the problem does not occur. Unfortunately, Weissinger's theory is not well founded theoretically, and one cannot be certain about its validity in unsteady flow. As will now be shown, this question can be answered by deriving a higher-order lifting-line theory using the matched asymptotic expansion method explained in §§ 3–7. Just a brief outline will be given, and further details may be found in Van Holten (1975*a*).

Substituting the first-order near field given by the first term on the right-hand side of (31) into (12) and rewriting this equation using (14), one obtains the following Poisson equation for p_{near}:

$$\frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \phi^2} = \frac{1}{A^2}\frac{l_1''}{\pi c}(\cosh\eta \sin\phi - \tfrac{1}{2}\sin 2\phi), \tag{44}$$

where l_1'' denotes the second derivative of the sectional lift l_1 with respect to the non-dimensional co-ordinate $z/\frac{1}{2}b$. In order to indicate that l_1'' is a function of the spanwise co-ordinate and to remain consistent with the other notation, we shall further use the notation $l_1''(z)$. This should *not* be interpreted as a derivative with respect to the argument z! The particular solution of (44) is given by

$$\frac{p}{\frac{1}{2}\rho U^2} = \frac{1}{A^2}\frac{C_{l_1}''(z)}{\pi}(\tfrac{1}{2}\eta \sinh\eta \sin\phi + \tfrac{1}{8}\sin 2\phi). \tag{45}$$

The complete solution of (44) may also contain solutions of the two-dimensional Laplace equation, of the form given by (17). The analysis outlined in §§ 3–7 can now be applied once again, leading to the following results:

$$p_{\text{near}} = -\frac{l_1(z)}{\pi c}\frac{\sin\phi}{\cosh\eta + \cos\phi} + \frac{1}{2A^2}\frac{l_1''(z)}{\pi c}(\eta \sinh\eta \sin\phi + \tfrac{1}{4}\sin 2\phi) + 2G(z)\cosh\eta\sin\phi. \tag{46}$$

$G(z)$ is the maximum difference in a plane z = constant between the pressure induced by a three-dimensional dipole line and the pressure of a two-dimensional dipole of equal local strength, measured at a point at a distance $\frac{1}{4}c$ from the line. According to (A 7), $G(z)$ has the value

$$G(z) = p_{\text{dip}}(\tfrac{1}{4}c, \tfrac{1}{2}\pi, z) + (2/\pi c)\, l_1(z), \tag{47}$$

where $p_{\text{dip}}(r, \chi, z)$ denotes the field of a dipole line, i.e. a harmonic field behaving for $r \to 0$ like

$$p_{\text{dip}}(r, \chi, z) \sim -l_1(z) \sin(\chi)/2\pi r. \tag{48}$$

The far-field behaviour of p_{near} according to (46), neglecting terms $O(A^{-3})$ and smaller, is

$$(p_{\text{near}})_{r\to\text{order }b} = -l_1(z)\frac{\sin\chi}{2\pi r} + l_1(z)\,\tfrac{1}{4}c\frac{\sin\chi}{2\pi r^2} + \frac{l_1''(z)}{2A^2\pi c}\left\{\frac{r}{\frac{1}{2}c}\ln\left(\frac{r}{\frac{1}{4}c}\right)\sin\chi + \tfrac{1}{4}\sin 2\chi\right\} + 2G(z)\frac{r}{\frac{1}{2}c}\sin\chi. \tag{49}$$

The far-field behaviour of p_{near} is seen to involve two singularities: a dipole as well as a quadrupole singularity. However, up to $O(A^{-2})$ the field (49) is equal to the field of a discrete dipole, shifted towards the quarter-chord position. Denoting cylindrical co-ordinates centred around the $\frac{1}{4}c$ line of the wing by (r', χ', z),

$$x = r'\cos\chi' - \tfrac{1}{4}c, \quad y = r'\sin\chi', \tag{50}$$

(49) may be written as

$$(p_{\text{near}})_{r\to\text{order }b} = -l_1(z)\frac{\sin\chi'}{2\pi r'} + \frac{l_1''(z)}{2A^2\pi c}\frac{r'}{\frac{1}{2}c}\ln\left(\frac{r'}{\frac{1}{4}c}\right)\sin\chi' + 2G(z)\frac{r'}{\frac{1}{2}c}\sin\chi'. \tag{51}$$

The composite pressure field to $O(A^{-2})$ is again built up as

$$p_{\text{composite}} = p_{\text{near}} + p_{\text{far}} - p_{\text{common}}, \tag{52}$$

where p_{near} is given by (46), p_{common} by (51) and p_{far} equals $p_{\text{dip}}(r', \chi', z)$, which denotes a dipole line along the quarter-chord line, behaving for $r' \to 0$ like

$$p_{\text{dip}}(r', \chi', z) \sim -l_1(z)\sin(\chi')/2\pi r' \quad \text{for} \quad r' \to 0. \tag{53}$$

In order to find the unknown function $l_1(z)$ we apply the tangency condition along the quarter-chord line (i.e. at $x = -\frac{1}{4}c$). The value of v/U at the quarter-chord line is found by transforming (32) into

$$\frac{v}{U}(-\tfrac{1}{4}c, 0, z) = \frac{1}{\rho U^2}\int_{-\infty}^{-\frac{1}{4}c}\frac{\partial p}{\partial y}(x, 0, z)\,dx, \tag{54}$$

which on substituting (52) yields

$$\frac{v}{U}(-\tfrac{1}{4}c, 0, z) = -\frac{C_{l_1}(z)}{2\pi} - \frac{v_i(z)}{U} - \frac{G(z)}{\rho U^2}, \tag{55}$$

where v_i is the induced velocity of classical lifting-line theory, defined as

$$\frac{v_i}{U}(z) = \frac{1}{\rho U^2}\int_{-\infty}^{-\frac{1}{4}c}\frac{\partial}{\partial y}\left\{p_{\text{dip}}(r', \chi', z) + l_1(z)\frac{\sin\chi'}{2\pi r'}\right\}_{y=0} dx. \tag{56}$$

On applying the tangency condition (55) yields after some rearrangement

$$C_{l_1}(z) = 2\pi\left\{\alpha(z) - \frac{v_i}{U}(z)\right\} - \frac{2\pi G(z)}{\rho U^2}. \tag{57}$$

The lift l_1 is the part of the total sectional lift associated with the first term on the right-hand side of (46). The total sectional lift is given by

$$C_l(z) = C_{l_1}(z) - 2\pi G(z)/\rho U^2. \tag{58}$$

10. The $\frac{3}{4}$-chord method in steady flow

Weissinger's $\frac{3}{4}$-chord method may be recovered from the set of equations given above by applying the tangency condition at the $\frac{3}{4}$-chord line (i.e. at $x = \frac{1}{4}c$) instead of applying it at the quarter-chord line. First of all, it should be remembered that near the wing surface the value of $p_{\mathrm{dip}}(r', \chi', z) - p_{\mathrm{common}}$ becomes zero, so that according to (51)

$$p_{\mathrm{dip}}(r', \chi', z) + l_1(z)\frac{\sin\chi'}{2\pi r'} = \frac{l_1''(z)}{2A^2\pi c}\frac{r'}{\frac{1}{2}c}\ln\left(\frac{r'}{c/4}\right)\sin\chi' + 2G(z)\frac{r'}{\frac{1}{2}c}\sin\chi'. \tag{59}$$

Therefore, integrating (59) up to the $\frac{3}{4}$-chord line and using (56) gives

$$\frac{1}{\rho U^2}\int_{-\infty}^{\frac{1}{4}c}\frac{\partial}{\partial y}\left\{p_{\mathrm{dip}}(r', \chi', z) + l_1(z)\frac{\sin\chi'}{2\pi r'}\right\}_{y=0} dx = \frac{v_i}{U}(z) + \frac{2G(z)}{\rho U^2} + \frac{C_{l_1}''(z)}{4\pi A^2}(\ln 2 - 1). \tag{60}$$

On solving (60) for v_i/U and substituting into (55), the tangency condition reads

$$-\alpha(z) = -\frac{C_{l_1}(z)}{2\pi} - \frac{1}{\rho U^2}\int_{-\infty}^{\frac{1}{4}c}\frac{\partial}{\partial y}\left\{p_{\mathrm{dip}}(r', \chi', z) + l_1(z)\frac{\sin\chi'}{2\pi r'}\right\}_{y=0} dx + \frac{G(z)}{\rho U^2} + \frac{C_{l_1}''(z)}{4\pi A^2}(\ln 2 - 1). \tag{61}$$

As may be seen from (60), the integral on the right-hand side of (61) is $O(A^{-1})$. We may then replace the function $l_1(z)$ in the integrand of (61) by $l(z)$ without introducing relevant errors, since according to (58) the difference between l_1 and l is a function $O(A^{-2})$. Furthermore, we may solve (58) for $C_{l_1}(z)$ and substitute this for the first term on the right-hand side of (61). Equation (61) thus transforms into

$$-\alpha(z) = -\frac{C_l(z)}{2\pi} - \frac{1}{\rho U^2}\int_{-\infty}^{\frac{1}{4}c}\frac{\partial}{\partial y}\left\{p_{\mathrm{dip}_l}(r', \chi', z) + l(z)\frac{\sin\chi'}{2\pi r'}\right\}_{y=0} dx + \frac{C_l''(z)}{4\pi A^2}(\ln 2 - 1), \tag{62}$$

where $p_{\mathrm{dip}_l}(r', \chi', z)$ denotes the field of a dipole line in the quarter-chord position, behaving for $r' \to 0$ like

$$p_{\mathrm{dip}_l}(r', \chi', z) \sim -l(z)\sin(\chi')/2\pi r', \tag{63}$$

i.e. its dipole strength distribution is based on $l(z)$ instead of on $l_1(z)$ as was p_{dip} in (53).

We now use Pistolesi's theorem (see e.g. Schlichting & Truckenbrodt 1967). This theorem states that the downwash at the $\frac{3}{4}$-chord point of a flat-plate aerofoil induced by a two-dimensional vortex at the $\frac{1}{4}c$-chord point having a strength which represents the lift of the aerofoil equals the correct downwash at the $\frac{3}{4}$-chord point. Therefore

$$-\frac{C_l(z)}{2\pi}-\frac{1}{\rho U^2}\int_{-\infty}^{\frac{1}{4}c}\frac{\partial}{\partial y}\left\{l(z)\frac{\sin\chi'}{2\pi r'}\right\}_{y=0}dx = 0, \tag{64}$$

and (62) finally reads

$$-\alpha(z) = -\frac{1}{\rho U^2}\int_{-\infty}^{\frac{1}{4}c}\frac{\partial}{\partial y}\{p_{\mathrm{d1p}_l}(r',\chi',z)\}_{y=0}\,dx+\frac{C_l''(z)}{4\pi A^2}(\ln 2-1). \tag{65}$$

Ignoring for a moment the last term, (65) states that the downwash induced along the $\frac{3}{4}$-chord line of a wing by a lifting line in the $\frac{1}{4}c$-position should equal the downwash required by the angle of incidence of the wing sections. We have thus recovered Weissinger's method. The last term in (65), which is $O(A^{-2})$, is the error in the $\frac{3}{4}$-chord method. However, the error vanishes for untwisted wings or wings with linear twist, since $C_l''(z)$ in the last term need not have better accuracy than $O(A^0)$ and may thus be based on a simple strip analysis. In more general cases the error may be expected to very small. Note that the proof of Weissinger's theorem is derived here only for a rectangular planform with uncambered sections.

11. Weissinger's method in unsteady flow

From the preceding section it has appeared that the central theorem needed to transform the higher-order lifting-line equations into Weissinger's formulation is Pistolesi's theorem. From this observation one can now conclude immediately that Weissinger's method does not remain valid in unsteady flow, because Pistolesi's theorem does not. Even without actually calculating the unsteady downwash at the $\frac{3}{4}$-chord point due to a variable two-dimensional vortex, this is clear from figure 6. The two-dimensional vortex representing a given function $l(t)$ causes a time-varying value of $v/U(t)$ at the $\frac{3}{4}$-chord point which is uniquely related to the function $l(t)$. In the case of the actual aerofoil, the function $v/U(t)$ and $l(t)$ are *not* uniquely related: their relation depends upon the particular motion considered, because of apparent-mass effects. The formulation of higher-order theory given in § 9 is much more suitable for extension of the theory to unsteady flow, and yields the complete unsteady pressure distribution on the wing surface. More details may be found in Van Holten (1975*a*), where the higher-order lifting-line theory, with appropriate modifications, is used to calculate the lift and moment distribution along helicopter blades in forward flight.

12. Conclusions

(i) The problem of singular induced velocities usually encountered in unsteady lifting-line theories is due to a misinterpretation of Prandtl's steady-flow model. A satisfactory and convenient definition of v_i can be derived.

(ii) Unsteady lifting-line theory can be formulated such that the section characteristics are expressed in terms of two-dimensional aerofoil characteristics. Three-dimensional effects should be taken into account by treating the variable induced velocity as a self-induced gust field.

(iii) Weissinger's $\frac{3}{4}$-chord method can be shown to be exact to $O(A^{-2})$ in the case of an uncambered, linearly twisted, rectangular wing in steady flow. The method is not usable in unsteady flow. However, a different formulation can be given to higher-order lifting-line theory which remains valid in unsteady flow. The latter theory yields the complete higher-order unsteady pressure distribution on the wing surface.

Appendix. The field of a dipole line

Writing the Laplace equation (3) in terms of the circular-cylinder co-ordinates (r, χ, z) (figure 3) and solving by separation (e.g. Moon & Spencer 1961), one finds that the pressure field must be built up from elementary solutions of the form

$$p(r,\chi,z) = \sin(n\chi)\, K_n(q[r/\tfrac{1}{2}b])\,\{A(q)\cos(q[z/\tfrac{1}{2}b]) + B(q)\sin(q[z/\tfrac{1}{2}b])\}, \quad \text{(A 1)}$$

where n and q are the separation constants and K_n denotes a modified Bessel function of the second kind. Only periodic solutions having antisymmetry with respect to $\chi = 0$ and having a singularity at $r = 0$ have been retained. As will be shown a line of dipoles with dipole strength $f(z/\frac{1}{2}b)$ is obtained by choosing

$$\left.\begin{aligned} n &= 1, \\ A(q) &= q\int_{-\infty}^{+\infty} f(\zeta/\tfrac{1}{2}b)\cos(q[\zeta/\tfrac{1}{2}b])\,d(\zeta/\tfrac{1}{2}b), \\ B(q) &= q\int_{-\infty}^{+\infty} f(\zeta/\tfrac{1}{2}b)\sin(q[\zeta/\tfrac{1}{2}b])\,d(\zeta/\tfrac{1}{2}b), \end{aligned}\right\} \quad \text{(A 2)}$$

and integrating (A 1) over all values of q between 0 and ∞:

$$p_{\text{dip}}(r,\chi,z) = \frac{\sin\chi}{2\pi}\frac{1}{\pi}\int_{-\infty}^{+\infty} f\left(\frac{\zeta}{\frac{1}{2}b}\right) d\left(\frac{\zeta}{\frac{1}{2}b}\right)\int_0^{\infty} qK_1\left(q\frac{r}{\frac{1}{2}b}\right)\cos\left(q\frac{\zeta - z}{\frac{1}{2}b}\right)dq. \quad \text{(A 3)}$$

The behaviour of p_{dip} near the line $r = 0$ is found by substituting the expansion

$$K_1(x) = \frac{1}{x} + \tfrac{1}{2}x\{\ln(\tfrac{1}{2}x) + (\gamma - \tfrac{1}{2})\} + \ldots, \quad \text{(A 4)}$$

and expanding the integral (A 3) asymptotically for $r/\frac{1}{2}b \to 0$. The first term of the asymptotic series is easily recognized as the Fourier integral representation of $[\sin\chi/2\pi(r/\frac{1}{2}b)]\, f(z/\frac{1}{2}b)$. Likewise, the second term is seen to have the structure of a gauge function $(\sin\chi/4\pi)\ (r/\frac{1}{2}b)\ \ln\ (r/\frac{1}{2}b)$ multiplied by the Fourier integral representation of the function $-f''(z/\frac{1}{2}b)$.

The expansion of (A 3) for $r \to 0$ thus reads

$$\begin{aligned} p_{\text{dip}}(r,\chi,z) = {} & \frac{\sin\chi}{2\pi(r/\frac{1}{2}b)} f\left(\frac{z}{\frac{1}{2}b}\right) - \frac{\sin\chi}{4\pi}\frac{r}{\frac{1}{2}b}\ln\left(\frac{r}{\frac{1}{2}b}\right) f''\left(\frac{z}{\frac{1}{2}b}\right) \\ & + \frac{\sin\chi}{4\pi}\frac{r}{\frac{1}{2}b} F\left(\frac{z}{\frac{1}{2}b}\right) + O\left\{\left(\frac{r}{\frac{1}{2}b}\right)^3 \ln\left(\frac{r}{\frac{1}{2}b}\right)\right\} \quad (r \to 0). \end{aligned} \quad \text{(A 5)}$$

As will be shown, there is no need to express $F(z/\frac{1}{2}b)$ in terms of $f(z/\frac{1}{2}b)$. For convenience, $\ln(r/\frac{1}{2}b)$, in the second term of (A 5), is rewritten as $\ln(r/\frac{1}{4}c) - \ln(2A)$, yielding the following alternative form of (A 5):

$$p_{\mathrm{dip}}(r,\chi,z) = \frac{\sin\chi}{2\pi(r/\frac{1}{2}b)} f\left(\frac{z}{\frac{1}{2}b}\right) - \frac{\sin\chi}{4\pi}\frac{r}{\frac{1}{2}b}\ln\left(\frac{r}{\frac{1}{4}c}\right) f''\left(\frac{z}{\frac{1}{2}b}\right) + \sin\chi\frac{r}{\frac{1}{2}b}2AG\left(\frac{z}{\frac{1}{2}b}\right) + \dots \quad \text{(A6)}$$

From (A 6) it follows that

$$G(z/\tfrac{1}{2}b) = p_{\mathrm{dip}}(\tfrac{1}{4}c, \tfrac{1}{2}\pi, z) - (A/\pi)f(z) + O(A^{-3}\ln A), \quad \text{(A 7)}$$

which provides us, using expression (19) for p_{dip}, with an efficient procedure to evaluate $G(z/\frac{1}{2}b)$ numerically to the required order of accuracy.

REFERENCES

Fung, Y. C. 1968 *An Introduction to the Theory of Aeroelasticity*. Dover.

James, E. C. 1975 Lifting-line theory for an unsteady wing as a singular perturbation problem. *J. Fluid Mech.* **70**, 753–771.

Lagerstrom, P. A. (ed.) 1967 *Fluid Mechanics and Singular Perturbations, A Collection of Papers by Saul Kaplun*. Academic.

Moon, P. & Spencer, D. E. 1961 *Field Theory for Engineers*. Van Nostrand.

Schlichting, H. & Truckenbrodt, E. 1967 *Aerodynamik des Flugzeuges*. Springer.

Van Dyke, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.

Van Holten, Th. 1974 Computation of aerodynamic loads on helicopter rotorblades in forward flight, using the method of the acceleration potential. *Proc. 9th Congr. Int. Counc. Aero. Sci., ICAS, Haifa, Israel* (ed. Robert R. Dexter & J. Singer), paper 74-54. Jerusalem: Weizmann Science Press of Israel.

Van Holten, Th. 1975*a* The computation of aerodynamic loads on helicopter blades in forward flight, using the method of the acceleration potential. *Delft Univ. Tech., Dept. Aerospace Engng, Rep.* no. VTH-189.

Van Holten, Th. 1975*b* On the validity of lifting line concepts in rotor analysis. *1st European Rotorcraft Powered Lift Aircraft Forum, Southampton*, preprint 20. (To be published in *Vertica, Int. J. Rotorcraft Powered Lift Aircraft*, vol. 1, (1976) no. 3.)